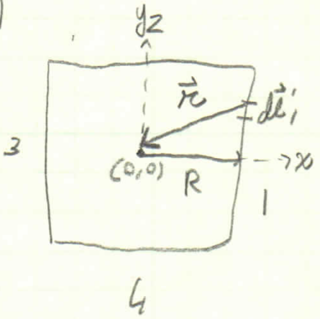
5.8

a)




$$B(\vec r) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\vec{\mathscr{r}} = -x\hat{x} - y\hat{y}$ $\quad \mathscr{r}^2 = x^2 + y^2$

$d\vec{l}'_1 = d\vec{l}_3 = dy\,\hat{y}$ $\quad \hat{\mathscr{r}} = \dfrac{-x\hat{x} - y\hat{y}}{\sqrt{x^2+y^2}}$

$d\vec{l}'_2 = d\vec{l}'_4 = dx\,\hat{x}$

$$dl_1 \times \hat{\mathscr{r}} = \frac{x\,dy}{\sqrt{x^2+y^2}}\hat{z}$$

$$dl_3 \times \hat{\mathscr{r}} = \frac{-y\,dx}{\sqrt{x^2+y^2}}\hat{z}$$

$$\int \frac{d\vec{l}'_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}\hat{z} = \int_{-R}^{R} \frac{x\,dy}{(x^2+y^2)^{3/2}}\hat{z} \text{ let } y = x\tan\theta \Rightarrow dy = x\sec^2\theta\, d\theta$$

$$\Rightarrow \theta_1 = \arctan\left(\frac{-R}{R}\right) = -\frac{\pi}{4}$$
$$\theta_2 = \arctan\left(\frac{R}{R}\right) = \frac{\pi}{4}$$

$$\frac{x\,dy}{(x^2+y^2)^{3/2}} = \frac{x\,(x\sec^2\theta)\,d\theta}{x^3(1+\tan^2\theta)^{3/2}} = \frac{d\theta}{x\sec\theta} = \frac{1}{x}\cos\theta\, d\theta$$

$$\int_{-R}^{R} \frac{x\,dy}{(x^2+y^2)^{3/2}}\hat{z} = \frac{1}{R}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta\, d\theta\,\hat{z} = \frac{\sin\theta}{R}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\hat{z} = \left(\frac{1}{R\sqrt{2}} - \frac{-1}{R\sqrt{2}}\right)\hat{z} = \frac{2}{R\sqrt{2}}\hat{z} = \frac{\sqrt{2}}{R}\hat{z}$$

$$\int \frac{d\vec{l}'_3 \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \int_{R}^{-R} \frac{x\,dy}{(x^2+y^2)^{3/2}}\hat{z} = \frac{\sin\theta}{-R}\Bigg|_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\hat{z} = \frac{\sin\theta}{R}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\hat{z} = \frac{\sqrt{2}}{R}\hat{z}$$

$$\int \frac{d\vec{l}'_2 \times \hat{\mathscr{r}}}{\mathscr{r}^2} = -\int_{R}^{-R} \frac{y\,dx}{(x^2+y^2)^{3/2}}\hat{z} = -\frac{\sin\theta}{R}\Bigg|_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\hat{z} = \frac{\sqrt{2}}{R}\hat{z}$$

$$\int \frac{d\vec{l}'_4 \times \hat{\mathscr{r}}}{\mathscr{r}^2} = -\int_{-R}^{R} \frac{y\,dx}{(x^2+y^2)^{3/2}}\hat{z} = \frac{\sin\theta}{R}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\hat{z} = \frac{\sqrt{2}}{R}\hat{z} \Rightarrow B(0,0) = \frac{\mu_0 I}{\pi R}\sqrt{2}\,\hat{z}$$

5.8 a) alternatively, using eq. 37,

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\hat{z}$$

For each segment $\theta_1 = -\frac{\pi}{4}$, $\theta_2 = \frac{\pi}{4}$, $s = R$

so

$$B = 4\frac{\mu_0 I}{4\pi R}\left(\sin\left(\tfrac{\pi}{4}\right) - S\left(-\tfrac{\pi}{4}\right)\right)\hat{z} = \frac{\mu_0 I}{\pi R}\left(\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\right)\hat{z} = \frac{\mu_0 I}{\pi R}\left(\frac{2}{\sqrt{2}}\right)\hat{z}$$

$$B = \frac{\mu_0 I}{\pi R}\sqrt{2}\,\hat{z}$$

b) for an n-sided polygon

$$\theta_1 = \frac{2\pi}{n}, \quad \theta_2 = \frac{-2\pi}{n}, \quad s = R$$

$$B_n = \frac{n\mu_0 I}{\pi R}\left[\sin\left(\frac{2\pi}{n}\right) - \sin\left(-\frac{2\pi}{n}\right)\right]\hat{z} = \frac{2n\mu_0 I}{\pi R}\sin\left(\frac{2\pi}{n}\right)\hat{z}$$

c) $$\lim_{n\to\infty} n\sin\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} \xrightarrow{\text{L'Hopital's Rule}} \frac{2\pi\cos\left(\frac{2\pi}{n}\right)\frac{d}{dn}\left(\frac{1}{n}\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} 2\pi\cos\left(\frac{2\pi}{n}\right) = 2\pi$$

so,

$$\lim_{n\to\infty} B_n = \frac{2\pi\mu_0 I}{4\pi R} = \frac{\mu_0 I}{2R}$$

from eq. (41), $$B(z=0) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+0)^{3/2}} = \frac{\mu_0 I}{2R}$$

so the equations indeed match!