2.24 Assuming $V = V(s, \phi)$, then $\nabla^2 V = 0$ becomes

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Let $V(s, \phi) = S(s)\Phi(\phi)$

Then

$$\nabla^2 V = \frac{\Phi}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{S}{s^2}\frac{\partial \Phi}{\partial \phi^2} = 0$$

$$s^2\frac{\nabla^2 V}{V} = \frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi}\frac{\partial \Phi}{\partial \phi^2} = C_1 + C_2 = 0$$

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) = \frac{s}{S}\left(\frac{\partial S}{\partial s} + s\frac{\partial^2 S}{\partial s^2}\right) = C_1$$

So then $\frac{\partial^2 S}{\partial s^2} + \frac{1}{s}\frac{\partial S}{\partial s} - \frac{S}{s^2}C_1 = 0$

This is an Euler-Cauchy equation, which is known to have a general solution given by

$S = C_1' s^{\lambda_1} + C_2' s^{\lambda_2}$ where λ_1 & λ_2 are given by the eq.'s characteristic equation:

$\lambda^2 + (\alpha - 1)\lambda + \beta = 0$ where here $\alpha = 1$ and $\beta = -C_1 \neq 0$

So $\lambda_{1,2} = \pm\sqrt{C_1}$. With this in mind we may choose a more convenient variable for C_1. Let $C_1 = k^2$.

Then $S(s) = C_1' s^k + C_2' s^{-k} = As^k + \frac{B}{s^k}$

The eq in ϕ is easy: $\Phi(\phi) = C_1''\cos(k\phi) + C_2''\sin(k\phi) = (D\cos k\phi + C\sin k\phi)$.

Then

$V(s, \phi) = \left(As^k + \frac{B}{s^k}\right)\left(C\sin(k\phi) + D\cos(k\phi)\right)$ for k an integer not equal to zero.

3.24

If $k=0$, the radial equation is not an euler-cauchy equation! Instead

$$\frac{d^2S}{ds^2}+\frac{1}{s}\frac{dS}{ds}=0. \quad \text{Let } P(s)=\frac{dS}{ds}.$$

Then $\frac{dP}{ds}=-\frac{1}{s}P \Rightarrow -\int\frac{dP}{P}=\int\frac{ds}{s}\Rightarrow -\ln(P)=\ln(s)+C$

$\Rightarrow P=\frac{C}{s}$ so $\frac{C}{s}=\frac{dS}{ds}\Rightarrow C\int\frac{ds}{s}=\int dS \Rightarrow S(s)=C\ln(s)+C'$

$\Rightarrow S(s)=E\ln(s)+F.$

Φ becomes $C\sin(0)+D\cos(0)=D$. So evidently for $k=0$, Φ is constant

so

$$V(S,\phi)=\begin{cases}E\ln(s)+F & \text{for } k=0\\ \left(As^k+\frac{B}{S^k}\right)\left(C\sin(k\phi)+D\cos(k\phi)\right) & \text{for } k>0\end{cases}$$

The most general solution is

$$V(s,\phi)=E\ln(s)+F+\sum_{k=1}^{\infty}\left(A_ks^k+B_ks^{-k}\right)\left(C_k\sin(k\phi)+D_k\cos(k\phi)\right)$$

But notice for $k\to -k$

$$\left(As^k+\frac{B}{S^k}\right)\left(C\sin(k\phi)+D\cos(k\phi)\right)\to\left(\frac{A}{S^k}+Bs^k\right)\left(D\cos(k\phi)-C\sin(k\phi)\right)$$

and the negative sign may be absorbed into the constant C.

Since A & B are arbitrary also, we see the $k\to -k$ simply maps $A_k\to B_k$ & $B_k\to A_k$. That is, $B_k=A_{-k}$ & vice versa. So the sum above,

$$\sum_{k=1}^{\infty}\left(A_ks^k+B_kS^{-k}\right)\left(C_k\sin(k\phi)+D_k\cos(k\phi)\right),$$ is equal to

$$\sum_{k=1}^{\infty}\left(A_kS^k+A_{-k}S^{-k}\right)\left(C_k\sin(k\phi)+D_k\cos(k\phi)\right)$$ which is equal to

$$\sum_{k=1}^{\infty}S^k\left(A_k\sin(k\phi)+B_k\cos(k\phi)\right)+\sum_{k=-1}^{\infty}S^k\left(C_k\sin(k\phi)+D_k\cos(k\phi)\right)$$

So $V(s,\phi)$ may also be written as

$$V(s,\phi)=A_0\ln(s)+B_0+\sum_{k=1}^{\infty}s^k\left(A_k\sin(k\phi)+B_k\cos(k\phi)\right)+\sum_{k=-1}^{-\infty}s^k\left(C_k\sin(k\phi)+D_k\cos(k\phi)\right)$$