3.16 The BC's are

(i) $V(0,y,z)=0$
(ii) $V(a,y,z)=0$
(iii) $V(x,0,z)=0$
(iv) $V(x,a,z)=0$
(v) $V(x,y,0)=0$
(vi) $V(x,y,a)=V_0$

inside the box $\{V\in\mathbb{R}^3 \mid x,y,z\in(0,a)\}$ and $\nabla^2V=0$

We assume V takes the form $V(x,y,z)=X(x)Y(y)Z(z)$ from which it follows

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_1,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2,\quad \frac{1}{Z}\frac{d^2Z}{dz^2}=C_3 \text{ and } C_1+C_2+C_3=0$$

let $C_3=k^2+l^2$ and $C_2=-k^2$

$$\text{So } X(x)=A\sin(lx)+B\cos(lx)$$
$$Y(y)=C\sin(kx)+D\cos(kx)$$
$$\text{and } Z(z)=Ee^{\sqrt{k^2+l^2}z}+Fe^{-\sqrt{k^2+l^2}z}$$

(i) & (iii) require, respectively, that $B=0$ & $D=0$

(ii) & (iv) require, respectively, that $l=\frac{n\pi}{a}$ & $k=\frac{m\pi}{a}$

(v) requires that $E+F=0\Rightarrow E=-F\Rightarrow Z(z)=2\sinh(\sqrt{k^2+l^2}\,z)$

So

$$V_{k,l}(x,y,z)=C_{k,l}\sin(lx)\sin(ky)\sinh(\sqrt{k^2+l^2}\,z)$$

or

$$V_{n,m}(x,y,z)=C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right)$$

and

$$V(x,y,z)=\sum_{n,m}V_{n,m}(x,y,z)=\sum_n\sum_m C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right)$$

with $V(x,y,a)=V_0$

3.16 To determine the coefficients we require a double integral

$$\int_0^a\int_0^a \sum_{n,m} C_{n,m} \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\pi\sqrt{n^2+m^2}\right) \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi x}{a}\right) dy\,dx$$

$$= C_{n',m'} \sinh\left(\pi\sqrt{n^2+m^2}\right) \int_0^a \sin^2\left(\frac{n'\pi x}{a}\right) dx \int_0^a \sin^2\left(\frac{m'\pi y}{a}\right) dy = \left(\frac{a}{2}\right)^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)$$

but

$$\left(\frac{a}{2}\right)^2 \sinh\left(\pi\sqrt{n^2+m^2}\right) C_{n,m} = V_0 \int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy\,dx = \left(\frac{a}{\pi}\right)^2 \frac{V_0}{nm} \Big[\cos(lx)\Big]_a^0 \Big[\cos(ky)\Big]_a^0$$

and $\left[\cos\left(\frac{n\pi x}{a}\right)\right]_a^0 \left[\cos\left(\frac{m\pi y}{a}\right)\right]_a^0 = \begin{cases} 0 & \text{if } n \text{ or } m \text{ even} \\ 4 & \text{if } n \text{ and } m \text{ odd} \end{cases}$

So

$$C_{n,m} = \begin{cases} 0 & \text{if } n \text{ or } m \text{ odd} \\ \dfrac{16V_0}{\pi^2 nm \sinh\left(\pi\sqrt{n^2+m^2}\right)} & \text{if } n \,\&\, m \text{ odd} \end{cases}$$

and so

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n,m\in 2\mathbb{N}-1}^{\infty} \frac{\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right)}{nm\sinh\left(\pi\sqrt{n^2+m^2}\right)} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \quad \square.$$