3.5 Consider a simply connected volume, $\mathcal{V}$, with charge density ρ. At each boundary of $\mathcal{V}$, either V or $\frac{\partial V}{\partial n} = \vec{\nabla} V \cdot \hat{n}$ is given. Then $\exists$ a unique solution to Poisson's equation in this region.

Proof: Let V_1 & V_2 denote two solutions to

$\nabla^2 V = \frac{1}{\varepsilon_0}\rho$ in the region specified.

Then $\nabla^2 V_1 = \frac{1}{\varepsilon_0}\rho$ and $\nabla^2 V_2 = \frac{1}{\varepsilon_0}\rho$

define $V_3 \equiv V_1 - V_2$. Then $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = \frac{1}{\varepsilon_0}\rho - \frac{1}{\varepsilon_0}\rho = 0.$

So V_3 satisfies not poisson's eq., but Laplace's equation.

At every boundary, either $V_1 = V_2$ and so $V_3 = 0$

or $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ meaning $\frac{\partial V_3}{\partial n} = \frac{\partial}{\partial n}(V_1 - V_2) = \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = 0$

Therefore, at every boundary V_3 is zero, or, $\frac{\partial V_3}{\partial n}$ is 0.

The case $V_3 = 0$ has been shown in the book. For $\frac{\partial V_3}{\partial n} = 0$:

$\vec{\nabla} \cdot (V_3 \vec{\nabla} V_3) = V_3 \nabla^2 V_3 + \vec{\nabla} V_3 \cdot (\vec{\nabla} V_3)$

and since $\nabla^2 V_3 = 0$ this leaves

$\vec{\nabla} \cdot (V_3 \vec{\nabla} V_3) = E_3^2$. Taking a closed volume integral of both sides,

$\oint_{\mathcal{V}} \vec{\nabla} \cdot (V_3 \vec{\nabla} V_3) d\tau = \oint_{\mathcal{V}} E_3^2 d\tau \Rightarrow \oint_S V_3 (\vec{\nabla} V_3) \cdot d\vec{a} = \oint_{\mathcal{V}} E_3^2 d\tau$

but $(\vec{\nabla} V_3) \cdot d\vec{a} = (\vec{\nabla} V_3) \cdot \hat{n}\, da$ and $\vec{\nabla} V_3 \cdot \hat{n} = \frac{\partial V_3}{\partial n} = 0$ so

$\oint_{\mathcal{V}} E_3^2 d\tau = 0$. This is only possible if $E_3 = 0$.

With $V_3 = V_1 - V_2$ $\vec{\nabla} V_3 = \vec{\nabla} V_1 - \vec{\nabla} V_2 \Rightarrow \vec{E}_3 = \vec{E}_2 - \vec{E}_1$

And hence $\vec{E}_2 = \vec{E}_1$ a unique $\vec{E}$ gaurantees a unique V. Since $V_1 = \int \vec{E}_1 \cdot d\vec{l} = \int \vec{E}_2 \cdot d\vec{l} = V_2$ $\square$.