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a) $V(\vec{r}) = \frac{q}{8\pi\varepsilon_0 R}\left[3-\left(\frac{r}{R}\right)^2\right]$; $\rho = \frac{q}{V} = \frac{q}{\frac{4}{3}\pi R^3}$

$$W = \frac{1}{2}\int_V \rho V d\tau \; ; \; \rho V = \frac{3q^2}{32\pi^2\varepsilon_0 R^4}\left[3-\left(\frac{r}{R}\right)^2\right]$$

$$W = \frac{\cancel{3/2}\, q^2}{32\pi^2\varepsilon_0 R^4}\int\left[3-\left(\frac{r^2}{R}\right)\right] r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$W = \frac{6\, q^2}{32\pi\varepsilon_0 R^4}\int_0^R \left[3r^2 - \frac{r^4}{R^2}\right] dr = \frac{3\, q^2}{16\pi\varepsilon_0 R^4}\left[r^3 - \frac{r^5}{5R^2}\right]_0^R$$

$$W = \frac{3q^2 R^3}{4^2\pi\varepsilon_0 R^4}\left[1-\frac{R^2}{5R^2}\right] = \frac{3q^2}{4^2\pi\varepsilon_0 R}\underbrace{\left[1-\frac{1}{5}\right]}_{4/5} = \frac{3q^2}{20\pi\varepsilon_0 R}$$

b) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau$; $E = \begin{cases} \frac{rq}{4\pi\varepsilon_0 R^3} & \text{for } 0<r<R \\ \frac{q}{4\pi\varepsilon_0 r^2} & \text{for } r>R \end{cases}$

$$\frac{2W}{\varepsilon_0} = k^2\int_0^R\left[\frac{r^2q^2}{R^6}\right]d\tau + k^2\int_R^\infty \frac{q^2}{r^4}d\tau \quad \text{let } k = \frac{1}{4\pi\varepsilon_0}$$

$$\frac{W}{2\pi\varepsilon_0 k^2 q^2} = \frac{1}{R^6}\int_0^R r^2 r^2 dr + \int_R^\infty r^{-2} dr = \frac{1}{R^6}\left[\frac{r^5}{5}\right]_0^R + \left[-r^{-1}\right]_R^\infty$$

$$\frac{W}{2\pi\varepsilon_0 k^2 q^2} = \frac{1}{R^6}\left[\frac{R^5}{5}\right] + \left[0+\frac{1}{R}\right] = \frac{1}{5}\cdot\frac{1}{R} + \frac{1}{R} = \frac{1}{R}\left(\frac{1}{5}+1\right) = \frac{6}{5R}$$

$$W = k^2\frac{12\pi\varepsilon_0 q^2}{5R} = \frac{1}{16\pi^2\varepsilon_0^2}\cdot\frac{12\pi\varepsilon_0 q^2}{5R} = \frac{3q^2}{20\pi\varepsilon_0 R}$$

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c) $W = \frac{\varepsilon_0}{2}\left(\int_{\mathcal{V}} E^2 d\tau + \oint_{S/\mathcal{V}} V\vec{E}\cdot d\vec{a}\right)$ $\quad d\vec{a} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$\frac{\varepsilon_0}{2}\int_{\mathcal{V}} E^2 d\tau = \frac{2\pi\varepsilon_0 k^2 q^2}{R^6}\left[\frac{r^5}{5}\right]_0^R + \left[-r^{-1}\right]_R^a 2\pi\varepsilon_0 k^2 q^2$$

$$= \frac{2\pi\varepsilon_0}{4^2\pi^2\varepsilon_0^2}\frac{q^2}{R}\left[\frac{1}{5}\right] - \left[\frac{1}{a} - \frac{1}{R}\right]\frac{2\pi\varepsilon_0}{4^2\pi^2\varepsilon_0^2} q^2$$

$$= \frac{q^2}{40\pi\varepsilon_0 R} - \frac{q^2}{8\pi\varepsilon_0 a} + \frac{q^2}{8\pi\varepsilon_0 R} = \frac{6q^2}{40\pi\varepsilon_0 R} - \frac{q^2}{8\pi\varepsilon_0 a}$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left[\frac{6}{5R} - \frac{1}{a}\right]$$

$\oint_{S/\mathcal{V}} V\vec{E}\cdot d\vec{a}$; $V = \frac{kq}{a}$

$$\vec{E}\cdot d\vec{a} = \frac{q}{4\pi\varepsilon_0}\left(\frac{a^2}{r^2}\right)\sin\theta \, d\phi \, d\theta \overset{r=a}{=} \frac{q}{4\pi\varepsilon_0}\sin\theta \, d\phi \, d\theta = kq\sin\theta \, d\phi \, d\theta$$

$$\underbrace{V(\vec{E}\cdot d\vec{a}) = \frac{(kq)^2}{a}\sin\theta \, d\theta \, d\phi}_{\text{not } \theta \text{ or } \phi \text{ dependent}}$$

$$\frac{\varepsilon_0}{2}\oint V(\vec{E}\cdot d\vec{a}) = 4\pi\varepsilon_0 \frac{k^2 q^2}{2a} = \frac{kq^2}{2a} = \frac{q^2}{8\pi\varepsilon_0 a}$$

$$W = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{6}{5R} - \frac{1}{a} + \frac{1}{a}\right] = \frac{3q^2}{20\pi\varepsilon_0 R}$$