1.59 $\vec{V} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$= 4r\sin\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(4r^2\cos\theta\sin\theta)$$

$$= 4r\sin\theta + \frac{4r^2}{r\sin\theta}(\cos^2\theta - \sin^2\theta) = 4r\sin\theta + 4r\frac{\cos^2\theta}{\sin\theta} - 4r\sin\theta$$

$$= 4r\cos^2\theta\csc\theta = 2r(1+\cos 2\theta)\csc\theta$$

In cylindrical coordinates a cone is given by

$Z = aS$ when $Z = R\cos 30$ $\quad S = R\sin 30$

so $a = \frac{S}{Z} = \tan(30°) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

$Z = \frac{S}{\sqrt{3}}$

but $\vec{\nabla}\cdot\vec{V}$ is in polar coordinates

$S = r\sin\theta$

$Z = r\cos\theta$, so the cone in polar coordinates is

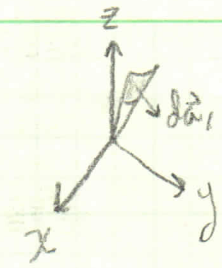
$\phi = \phi$ $\qquad r\cos\theta = \frac{r\sin\theta}{\sqrt{3}} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30$

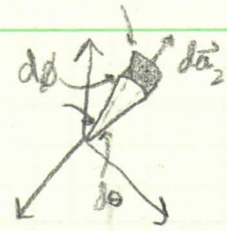
So in polar coordinates, a cone is a surface created by fixing θ
this is called a coordinate surface.

$$\int_V \vec{\nabla}\cdot\vec{V}\,d\tau = \int_0^R\int_0^{2\pi}\int_0^{\pi/6} 4r^3\cos^2\theta\csc\theta\,\sin\theta\,d\theta\,d\phi\,dr = \frac{1}{2}R^4\pi\int_0^{\pi/6}(1+\cos 2\theta)\,d\theta$$

$$= R^4\pi\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} = R^4\pi\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] = \frac{\pi R^4}{12}\left[2\pi + 3\sqrt{3}\right]$$

$$\int_V \vec{\nabla}\cdot\vec{V}\,d\tau = \oint_S \vec{V}\cdot d\vec{a}$$





2 surfaces:

Cone:

$$d\vec{a}_1 = dr\, r\sin\theta\, d\phi\, \hat{\theta}$$

$$\vec{V}\cdot d\vec{a}_1 = 4r^3\cos\theta\sin\theta\, dr\, d\phi$$

$$\int_0^R\int_0^{2\pi} 4r^3\cos\theta\sin\theta\, dr\, d\phi = 2\pi R^4\sin(\pi/3) = \sqrt{3}\,\pi R^4$$

sphere: $d\vec{a} = r^2\sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\vec{V}\cdot d\vec{a} = \frac{R^4}{2}(1-\cos(2\theta))\, d\theta\, d\phi$$

$$\frac{R^4}{2}\int_0^{\pi/6}\int_0^{2\pi}(1-\cos 2\theta)\, d\phi\, d\theta = \pi R^4\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = \pi R^4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} - 0 + 0\right)$$

$$\oint_S \vec{V}\cdot d\vec{a} = \pi R^4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{4\sqrt{3}}{4}\right) = \pi R^4\left(\frac{2\pi}{12} + \frac{\sqrt{3}}{4}\right) = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$