1.57 $\vec{V} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$

starting at origin,

Path (i) $\phi = 0,\ \theta = \frac{\pi}{2},\ d\vec{l} = dr\,\hat{r}$

$\vec{V}\cdot d\vec{l} = r\cos^2\theta\, dr = r\,dr$

$\int_0^1 r\,dr = \frac{1}{2}$

Path (ii) $r = 1,\ \theta = \frac{\pi}{2},\ d\vec{l} = r\,d\phi\,\phi$

$\vec{V}\cdot d\vec{l} = 3r^2 d\phi = 3d\phi$

$\int_0^{\frac{\pi}{2}} 3d\phi = \frac{3\pi}{2}$

Path (iii) $\phi = \frac{\pi}{2};\ d\vec{l} = dz\,\hat{z}$ $z = r\cos\theta \Rightarrow dz = \cos\theta\,dr - r\sin\theta\,d\theta$

$\vec{V}\cdot d\vec{l} = \vec{V}\cdot\hat{z}\,z$; $\vec{V}\cdot\hat{z} = r\cos^3\theta + r\cos\theta\sin^2\theta$; $r\sin\theta = 1 \Rightarrow z = \cot\theta$

$r = (\sin\theta)^{-1} \Rightarrow dr = -(\sin\theta)^{-2}\cos\theta\,d\theta = -\frac{\cos\theta}{\sin^2\theta}d\theta$

$\Rightarrow dz = -\frac{\cos^2\theta}{\sin^2\theta}d\theta - d\theta = -d\theta(\cot^2\theta + 1) = -d\theta\csc^2\theta$

$\vec{V}\cdot d\vec{l} = -\left(\frac{\cos^3\theta}{\sin\theta} + \cos\theta\sin\theta\right)\frac{d\theta}{\sin^2\theta} = -(\cot^3\theta + \cot\theta)d\theta = -\cot\theta\csc^2\theta\,d\theta$

$-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta\,d\theta}{\sin^3\theta}$ let $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta \Rightarrow -\int_1^{\frac{1}{\sqrt{2}}} u^{-3}du = \frac{1}{2}u^{-2}\Big|_1^{\frac{1}{\sqrt{2}}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

Path (iv) $\theta = \frac{\pi}{4},\ \phi = \frac{\pi}{2}$ $r: 1 \to 0$

$d\vec{l} = dr\,\hat{r}$

$\vec{V}\cdot d\vec{l} = r\cos^2\theta\,dr = \frac{r}{2}dr$

$\int_1^0 \frac{r}{2}dr = \frac{r^2}{4}\Big|_1^0 = -\frac{1}{4}$

So $\oint \vec{V}\cdot d\vec{l} = \frac{1}{2} + \frac{3\pi}{2} - \frac{1}{4} - \frac{1}{4} = \frac{3\pi}{2}$

1.57

$$\oint_P \vec{v}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

recognizing at the Surface in the xy plane, $\theta = \frac{\pi}{2}$, $d\vec{a} = r\,d\phi\,dr\,\hat{\theta}$

we only need the $\hat{\theta}$ term in the curl

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cancel{(r\cos^2\theta)} - \frac{\partial}{\partial r}(r\,3r)\right] = \frac{1}{r}\left[0 - 6r\right] = -6$$

$$\int_0^1\int_0^{\frac{\pi}{2}} 6r\,d\phi\,dr = \frac{3\pi}{2}$$

for the surface in the yz plane, $\phi = \frac{\pi}{2}$, $d\vec{a} = r\,d\theta\,dr\,\hat{\phi}$

so we only need the $\hat{\phi}$ term of the curl

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi} = 0$$

$$\int_{S_2}(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0$$