1.38 a unit vector points in the direction the vector increases. If vector $\vec{r}$ is a function of several variables (call one u), then by definition of the partial derivative, the vector

$$\vec{e}_u = \frac{\partial \vec{r}}{\partial u} \qquad (1)$$

will point in the direction of increasing u.

Any vector is 3-space can be written as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \text{ in catesian coordinates,} \qquad (2)$$

or

$$\vec{r} = r\hat{r} \text{ in spherical coordinates.} \qquad (3)$$

However, $\hat{r}$ has dependence of θ & ϕ, to be explicit

$\vec{r} = r\hat{r}(\theta, \phi)$. This is different from the carterion setting in which the direction of $\hat{\imath}, \hat{\jmath}$, & $\hat{k}$ do not change.

Thus, from (1) & (3)

$$\vec{e}_r = \frac{\partial \vec{r}}{\partial r} = \hat{r}(\theta, \phi) = \frac{\partial x}{\partial r}\hat{\imath} + \frac{\partial y}{\partial r}\hat{\jmath} + \frac{\partial z}{\partial r}\hat{k}$$

$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = r\frac{\partial \hat{r}(\theta, \phi)}{\partial \theta} = \frac{\partial x}{\partial \theta}\hat{\imath} + \frac{\partial y}{\partial \theta}\hat{\jmath} + \frac{\partial z}{\partial r}\hat{k}$$

$$\vec{e}_\phi = \frac{\partial \vec{r}}{\partial \phi} = r\frac{\partial \hat{r}(\theta, \phi)}{\partial \phi} = \frac{\partial x}{\partial \phi}\hat{\imath} + \frac{\partial y}{\partial \phi}\hat{\jmath} + \frac{\partial z}{\partial \phi}\hat{k}$$

We know that
$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

So from (2) $\vec{r} = r\cos\phi\sin\theta\hat{\imath} + r\sin\phi\sin\theta\hat{\jmath} + r\cos\theta\hat{k} \qquad (4)$

Combining (3) & (4) then $\hat{r} = \frac{\vec{r}}{r} = \cos\phi\sin\theta\hat{\imath} + \sin\phi\sin\theta\hat{\jmath} + \cos\theta\hat{k}$

1.38 so in cartesian coordinates,

$$\vec{e}_\theta = r\frac{\partial \hat{r}(\theta,\phi)}{\partial\theta} = r\cos\phi\cos\theta\,\hat{\imath} + r\sin\phi\cos\theta\,\hat{\jmath} - r\sin\theta\,\hat{k}$$

and

$$\vec{e}_\phi = r\frac{\partial \hat{r}(\theta,\phi)}{\partial\phi} = -r\sin\phi\sin\theta\,\hat{\imath} + r\cos\phi\sin\theta\,\hat{\jmath}$$

all that's left then is to normalize each e_u, and they will be unit vectors.

$$\hat{r} = \frac{\vec{e}_r}{|\vec{e}_r|}; \quad \hat{\theta} = \frac{\vec{e}_\theta}{|\vec{e}_\theta|}; \quad \hat{\phi} = \frac{\vec{e}_\phi}{|\vec{e}_\phi|}$$

$$|\vec{e}_u| = \sqrt{\vec{e}_u \cdot \vec{e}_u}$$

$$\vec{e}_r \cdot \vec{e}_r = \cos^2\phi\sin^2\theta + \sin^2\phi\sin^2\theta + \cos^2\theta = (\cos^2\phi + \sin^2\phi)\sin^2\theta + \cos^2\theta = 1$$

so $\hat{r} = \dfrac{\vec{e}_r}{1} = \cos\phi\sin\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}$

$$\vec{e}_\theta \cdot \vec{e}_\theta = r^2\cos^2\phi\cos^2\theta + r^2\sin^2\phi\cos^2\theta + r^2\sin^2\theta = r^2\left[(\cos^2\phi + \sin^2\phi)\cos^2\theta + \sin^2\theta\right]$$

so

$\hat{\theta} = \dfrac{\vec{e}_\theta}{r} = \cos\phi\cos\theta\,\hat{\imath} + \sin\phi\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{k}$

$$\vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\sin^2\theta = r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)$$

so

$\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$