# 4 Spatial Dimension Chess

Perhaps surprisingly, I have had many of my friends ask me about chess in higher dimensions, or muse about such an idea. Having been posed this question so often,I've decided to write up this article explaining how one might extend the board, pieces, and rules of chess into the third and fourth spatial dimensions (henceforth called 3-space and 4-space respectively or $\mathbb{R}^3$ and $\mathbb{R}^4$ when appropriate).

## 1 Constructing a Play Field in Higher Dimensions

The extension of the chess board into 3-space is rather straight forward: from a square to a cube. Since the chess board is an 8 by 8 arrangement of squares, a chess cube would be an 8 by 8 by 8 arrangement of cubes. This choice is rather arbitrary of course; we could simply stacks two chess boards one above the other, and as long as the pieces may move between boards, we have a valid 3-space chess board. And indeed, creating a bona fide chess cube creates quite a bit of free space for the meager, sixteen-piece armies to roam inside a cube with 512 spaces (compared to the 64 spaces in standard chess, or "chess classic"). This problem becomes yet again amplified to exponential extremes in 4-space where an 8 by 8 by 8 by 8 hypercube offers $8^4 = 4096$ spaces for the 32 pieces to occupy! We'll come back to this problem later, but for now it will suffice to assume that the play field offers a dimension of movements equal to the dimension in which it is played.

The idea of a chess cube is nothing too particularly mind boggling. Rather than the pieces resting on squares, they rest inside cubes, or bins. If we were to look at the play field from a bird's eye view, it would simply look like a standard chess board[1]. We could continue moving down, removing the top layer of cubes slice by slice, and each time we see another 2D chess board. In a sense, to a 2D inhabitant, it's as if there are eight 2D boards all seemingly existing "inside" of each other.

Now Consider the following: We can start with 64 cubes and arrange them in a square. Then, we can take another 64 cubes, and stack each above a unique cube from the first row. We can continue on in this manner until all the layers we want have been added and the chess cube is complete. But let's build it a slightly different way. We'll begin with an ideal 2D chess board (ideal meaning the squares have no thickness—they're purely 2 dimensional). Next, lay a second 2D chess board not above the first, but to the right, leaving a small gap so as to keep the boards separate. Now lay a third down in the same manner.

[1] perhaps with the colors swapped, but in this article we need not concern ourselves with the colors of spaces which are merely for the convenience of the player

And a fourth, and so on until a set of 2D chess boards all arranged in a line. Rather than thinking about the chess pieces moving up and down our previously constructed cube, we can think about the pieces moving to the same square on the board to the right or left of it. In this manner we have found a way to play 3-space chess in 2-space.

With these ideas in mind, grappling with the notion of a chess hypercube and its construction is somewhat easier to comprehend. To us 3-space inhabitants, we may think of a chess hypercube as a number of chess cube boards coexisting inside of each other. And the movement along the 4th dimension is the movement from one board to the next. In similar fashion, we may construct a play field in 3-space by placing chess cubes side-by-side, and interpret the movement along the 4th dimension as jumping to the same space on an adjacent cube.

# 2 Positioning and Movement in Higher Dimensions

But how might the pieces move inside a chess cube or hypercube? I believe the movement of the pieces can be most naturally extended to higher dimensions with the help of some linear algebra. Let's attach a coordinate frame to the bottom left most side of a 2D chess board (on the bottom left of the a1 tile). Call the axis extending along the rows $x_1$ and the axis extending along the columns $x_2$. We can then use position vectors to denote the space occupied by a piece, and displacement vectors to describe the movement of pieces around the board. Let $i, j, k$ be natural numbers less than or equal to the dimension of the chess board. Let the unit vector of axis $x_i$ be denoted by $\hat{x_i}$ where $\hat{x_i}$ is a row vector with the same length as the chess board's dimension, and all zeroes except in the ith position which contains a 1. We can interpret these unit vectors to mean "move one space in the $x_i$ direction". Then the set of unit vectors, $\{\hat{x_i}\}$ , forms a complete basis for describing the movement of pieces. As an example, consider white's queen-side rook in its starting position in a game of 2D chess. It's position vector is described by the vector $\hat{x_1}+\hat{x_2}$. Suppose at some later time we want to move the rook 3 spaces forward. We can describe this by the displacement vector $3\hat{x_2}$. Adding this to the Rook's position vector we have $\hat{x_1} + \hat{x_2} + 3\hat{x_2} = \hat{x_1} + 4\hat{x_2}$ which is the space the rook will occupy after being moved.

## 2.1 Movement in 2-Space

I'll begin by describing the movements of the pieces on a 2D chess board and then show how they might be extended naturally to higher dimensions. In 2-space our basis is $\{\hat{x_1},\hat{x_2}\}$. Let's start with the rook[2]. The rook's movement can

[2] I am going to assume here that anyone still reading this already understands how the chess pieces move.

be described as $(\pm r)\hat{x_1}$ or $(\pm r)\hat{x_2}$ where r is the number of spaces the rook has moved. That is, the rook's movement is the set $R = \{(\pm r)\hat{x_i}\}$. The bishop's movement can be described as $(\pm b)(\hat{x_1} \pm \hat{x_2})$ where b is the number of spaces the bishop has moved[3]. Denoting the set of bishop's movements as B, we can more generally write the set as $B = \{\ (\pm b)(\hat{x_i} \pm \hat{x_j})\}$ where i and j are index values and $i \neq j$. The queen's movement then can be denoted by the set $Q = R \cup B$. The knight is slightly more tricky to pin down, but in 2-space we can see that it is given by $(\pm 1) * (2\ \hat{x_1} \pm \hat{x_2}\ )$ or $(\pm 1) * (\ \hat{x_1} \pm 2\ \hat{x_2}\ )$ so the knight's movements are given by the set $N = \{(\pm 1) * (2\ \hat{x_i} \pm \hat{x_j}\ )\}$ again with $i$ not equal to $j$. The king's possible movements may be thought of as a special case of the queen's movements with $r = b = 1$. That is $K = \{v \in Q | r = b = 1\}$. Last is the pawn, the only piece to have a different set of movements for capturing. We can denote the set of capturing movements as $P_c$ and the set of non-capturing movements as $P_m$ and the set of all possible pawn movements as $P = P_c \cup P_m$. Then $P_c = \{(-1)^b \hat{x_2} \pm \hat{x_1}\}$ where $b = 0$ for white, 1 for black and $P_m = \{(-1)^b 2\ \hat{x_2}\ , (-1)^b\ \hat{x_2}\ \}$. $P_c$ may be generalized to $P_c = \{(-1)^b \hat{x_2} \pm \hat{x_i})\}$ where $i \neq 2$ and the same restriction applies to b[4].

## 2.2 Movement in 3-Space

Now we are in a good position to extend the movements of pieces into 3-space. And in fact, as generalized as things are, there isn't much more that needs to be added to each set. I'll begin by giving a table of what must be added, if anything, to the pieces' movements, and then provide my justifications afterwards. Remember, now that we are in 3-space, $i, j, k$ can take on the values 1, 2, or 3.

| **Piece** | **Additional movements** |
|---|---|
| Rook | No additions needed |
| Bishop | Now $(\pm b)(\ \hat{x_i} \pm\ \hat{x_j}\ \pm\ \hat{x_k}\ )$ with $i, j, k$ unique |
| Queen | No additions needed |
| King | No additions needed |
| Knight | No additions needed |
| Pawn | add to $P_c$ the movements $\pm\ \hat{x_i} \pm\ \hat{x_2}\ \pm\ \hat{x_j}$ where the sign of $\hat{x_2}$ is as usual, $i, j \neq 2$ and unique |

**Rook:** The characteristic movement of the rook is that it moves along a single axis.

**Bishop:** One of 2 pieces that explicitly needs additions, we must accommodate the 8 new diagonals a bishop can move along, which are characterized as an equal number of steps along all axes. In fact, in 2-space that *is* how the bishop

[3]At this point we don't necessarily need to assume the pieces are making valid moves, but we should expect anyone playing chess is following the rules.

[4]Remember, I'm making no claim to the legality of these moves during play, but simply stating they are *possible* moves of the pieces

moves: an equal number of spaces horizontally and vertically. It could be argued that this addition should actually be the only movements available to the bishop. But I disagree with this notion for two reasons. First, the fact that in 2-space the bishop moves an equal amount along all axes is an artifact of the space it lives in—there can't be diagonal movement any other way. It's mistaking the weak claim of diagonal movement with a much stronger claim. In higher dimensional spaces, we can still achieve diagonal movement without needing to move along all axes simultaneously, and I believe that bishops moving diagonally, however achieved, is at the heart of the piece's nature. The second, albeit weaker, argument is that forcing this restriction severely limits the mobility of the piece. Since this is an argument dealing with game balance, I'll hold off any further comments until later.

**Queen:** The queen is still the set of any possible movements made by a rook or bishop. With the rook and bishop properly extended to 3-space, so too is the queen.

**King:** The king is a special case of the queen's movement with $r = b = 1$, and so naturally follows.

**Knight:** Although no additional moves are needed for the knight, I encourage the reader to stop and consider the new possibilities of movement opened up by the addition of a third dimension. As an exercise, consider the difference between $i = 2, j = 3$ and $i = 3, j = 2$.

**Pawn:** This is easily the most difficult piece to extend into higher dimensions, due to the unusual characteristics only it posses. Along with having a different set of movements while capturing, it is the only piece not permitted to move "backwards". But the concept of backwards itself becomes murky when speaking in higher dimensions. I have interpreted it to mean that the pawn's displacement vector may not have a negative $\hat{x_2}$ component if playing white, nor a positive $\hat{x_2}$ component if playing black. I have also chosen to not change or add to its non-capturing move set. This is probably the most dubious choice to be made, but my justification is as follows: In 2-space the pawn is locked into a single rank that it can only move away from by capturing. With the pawn's movements in 3-space defined as they are, this idea of the pawn only being able to change it's rank by capturing is extended to 3-space: we have locked the pawn not only into its rank, but also onto its board, only being able to move up or down through capture as well.

To help with grasping how the pawn captures in 3-space, here's a visualization. The spaces available to the pawn for capture form a ring circling around the space $+(-1)^b\hat{x_2}$ away from the pawn.

## 2.3 Movement in 4-Space

I think by this point the pattern for extension into 4-space is fairly straightforward. As before, I'll start with the table and make my comments afterwards.

| Piece | Additional movements |
|---|---|
| Rook | No additions needed |
| Bishop | $(\pm b)(\ \hat{x_1} \pm \hat{x_2} \pm \hat{x_3} \pm \hat{x_4}\ )$ |
| Queen | No additions needed |
| King | No additions needed |
| Knight | No additions needed |
| Pawn | add to $P_c$ the movements $\pm \hat{x_1} \pm \hat{x_2} \pm \hat{x_3} \pm \hat{x_4}$ where the sign of $\hat{x_2}$ is as usual |

**Bishop:** This is including in the set the possibility of movement along all 4 dimensions simultaneously. This is analogous to the extension made from 2- to 3-space.
**Pawn:** Essentially the same argument as the bishop.
With this, we have fully define the set of all possible movements for all pieces in 4-space. Now we must extended the rules to 4-space, and of course make sure checkmating the king is still possible.

## 2.4 Formalism

Up to this point the position and movement vectors have been given as explicit sums of scaled basis vectors (i.e. a linear combination of the basis vectors). Now I will introduce a notational convention for condensing these vectors and specifying a board layout.
There are of course many ways to notate vectors, and for the easy of typing, I shall choose the method of using a chevron of ordered numbers: $< a, b, c, d >$, where a, b, c, and d correspond to the scalar multiples of $\hat{x_1}$,$\hat{x_2}$,... respectively. For example, the position 3 $\hat{x_1}$ + $\hat{x_2}$ +3 $\hat{x_3}$ +3 $\hat{x_4}$ may simply be written as $< 3, 1, 3, 3 >$. And of course, addition follows in the usual way: $< a, b, c, d >$ $+ < e, f, g, h > = < a + e, b + f, c + g, d + h >$.
Borrowing the abbreviations for the pieces as designated by FIDE, we can then specify a piece's position by first writting its abbreviation, then its position vector. For example, at the beginning of a 2D chess game, the white queen's position may be specified as Q$< 4, 1 >$. We need then to create a table listing all pieces and their position for black and white. The following table gives how the starting position of 2D chess could be notated:

| **White** | **Black** |
|---|---|
| P$< 1,2 >$ | P$< 1,7 >$ |
| P$< 2,2 >$ | P$< 2,7 >$ |
| P$< 3,2 >$ | P$< 3,7 >$ |
| P$< 4,2 >$ | P$< 4,7 >$ |
| P$< 5,2 >$ | P$< 5,7 >$ |
| P$< 6,2 >$ | P$< 6,7 >$ |
| P$< 7,2 >$ | P$< 7,7 >$ |
| P$< 8,2 >$ | P$< 8,7 >$ |
| R$< 1,1 >$ | R$< 1,8 >$ |
| R$< 8,1 >$ | R$< 8,8 >$ |
| N$< 2,1 >$ | N$< 2,8 >$ |
| N$< 7,1 >$ | N$< 7,8 >$ |
| B$< 3,1 >$ | B$< 3,8 >$ |
| B$< 6,1 >$ | B$< 6,8 >$ |
| Q$< 4,1 >$ | Q$< 4,8 >$ |
| K$< 5,1 >$ | K$< 5,8 >$ |

# 3 Extending the Rules

Ideally we should change as little as possible. And indeed there really isn't anything that needs to be changed. Perhaps however, some of the rules involving pawns needs to be made more explicit. For en passant captures we state the rule as follows:

**Definition 1. En Passant:** When a pawn moves two spaces, and in so doing passes through a space an opponent pawn can capture, the opponent pawn may move to the skipped-over space and capture the pawn. The en passant capture must be made on the very next turn or the right to do so is lost.

This isn't really changing the rule for en passant captures at all, but makes it more clear how it is handled in n-space. Next we must make more clear under what circumstances a pawn may promote, which I shall define as thus:

**Definition 2. Pawn Promotion:** A white pawn must promote if the $\hat{x_2}$ component of its position vector is equal to 8. A black pawn must promote if the $\hat{x_2}$ component of its position vector is equal to 1. As always, the pawn may not promote to a king or pawn, and must promote to a piece of its own color.

This continues with the idea of trying to preserve a notion of "forwards" and "backwards" across the board in higher dimensions.
We can in a sense inherit the rest of chess rules as they are in 2-space without any more need for clarification; however, as shall be discussed later, one may wish to tweak the rules in various ways to facilitate balance in higher dimensions.
Without any further adjustments, it remains to be shown that the rather strict requirement of checkmate can still occur in higher dimensions. As the well

versed reader knows, there are many different combinations of pieces that can checkmate the king: king and queen, king and rook, rook and rook, rook and queen, bishop and knight, etc. The queen and king is amongst strongest combinations so we will begin by examining this combination in higher dimensions. Although there are of course exceptions in play, checkmate typically breaks down into two components: first force the king to the edge of the play field (to cut down on his possible movements), then position the pieces such that the king is checkmated. We'll begin first by assuming that the king is already on the edge of the board. Specifically let us consider the following board layout with black to move:

| **White** | **Black** |
|---|---|
| Q$< 3,2,3,3 >$ | |
| K$< 3,3,3,3 >$ | K$< 3,1,3,3 >$ |

Assuming white moved the queen last, and it was not attacking black's king before the move, the reader can quickly verify that this is indeed a valid layout. First let us notice that black's king is in check. Since black's only piece left on the field is the king, black must move the king. Let $r_K$ be the displacement vector representing Black's king's move. If Black's king makes the specific movement $r_K =< 0,1,0,0 >$, capturing the queen, then white's king can capture with the movement $< 0,-1,0,0 >$. So, this is not a valid move because it leave's black's king in check. For any other move made by the king, the queen can capture by making the move $r_K + < 0,-1,0,0 >$ since the $\hat{x_2}$ component of $r_K$ cannot be less than zero. Thus, Black's king has no legal moves. Since the king is also in check, this position is indeed a checkmate! It is easy to generalize this position: Let $p_K$ be the position of black's king. If at least one of the components of $p_K$ is equal to 1, then the following layout is checkmate:

| **White** | **Black** |
|---|---|
| Q$p_K + \hat{x_i}$ | |
| K$p_K + 2\ \hat{x_i}$ | K$p_K$ |

where i is the index of a component of $p_K$ equal to unity. We can strengthen this mating position by noting that more pieces may be on the board, so long as none of Black's piece's are attacking White's queen.

It's well known that in 2-space chess, King+Queen v. King is a winning position for the player with the Queen regardless of thie piece positions (assuming a valid configuration of course). That is, no matter where the kings and queen are, the player with the queen can always make a sequence of moves that leads to checkmate. Is the same true in 4-space?. Let's consider how the mating algorithm works in 2-space, and without loss of generality, let's assume white as the queen. The trick for white is to maintain "opposition". That is, the queen is always positioned such that the black king is forced towards an edge of the board. For instance, if black's king is on $< 2,2 >$, then white can place the queen on $< 3,4 >$ and black's king is forced to move closer to an edge. The trick is to "box-in" the king. This trick doesn't work in 4-space though[5]!

[5] With the pieces movements defined as I have done so

suppose Black's king occupies some arbitrary position, $p_K$. Placing a queen at the position $p_K + 1\hat{x_i}+2\hat{x_j}, i \neq j$ no longer works—the extra dimensions of 4-space have opened up spaces for the king to escape to!

# 4 Practial Considerations

## 4.1 Fugacity

Let $R_n$ denote the ratio of pieces on the board to total number of spaces on the board for dimension n, which I have named the "fugacity of n-space". The fugacity of 2-space at the beginning of a game is $R_2 = 32/64 = 0.5$ If we do not add more pieces and use an 8x8x8x8 board, the fugacity of 4-space is $R_4 = 32/8^4 = 1/256 \approx 0.0078$. Going from 2- to 4-space has caused a 3 order of magnitude reduction in the fugacity of the space. The minimum possible fugacity of 2-space is $R_2 = 2/64 = 0.03125$; still an order of magnitude higher than the starting position in 4-space. While true it hasn't been shown that the fugacity is in any way a meaningful metric, it does give a hint to the spaces available for occupation by the pieces, with a lower fugacity suggesting more spaces available. Furthermore, it also roughly hints at how likely it is a piece is within capturing range of another. Imagine the chess pieces like particles of a gas in a container. A low fugacity suggests a low frequency of collisions, and so if the pieces begin clustered together, they will quickly disperse and rarely "see" each other. Further testing is still needed to see if such a low starting position fugacity of 4-space is indeed problematic, but it is this author's opinion after initial testing that, the fugacity must be raised higher for an interesting game to develop, and suggests the following modification. Play takes place not on an 8x8x8x8 board, but an 8x8x2x2 board. Instead of beginning with 16 pieces, each side begins with 61 pieces: quadruble of all pieces in a 2-space game except the king, of which there remains just one. This raises the fugacity to $R_4 = 122/8^2/4^2 \approx 0.119$

## 4.2 Starting Position

Let us disregard for the moment the modifications previously suggested. Suppose instead we remain with the 8x8x8x8 board and 16 pieces per side. How are the pieces to be arranged on the board? It seems most natural to simply pick a plane in the board and arrange the pieces as in 2-space chess. The author suggests the following arrangement:

| **White** | **Black** |
|---|---|
| P$< 1,2,5,5 >$ | P$< 1,7,5,5 >$ |
| P$< 2,2,5,5 >$ | P$< 2,7,5,5 >$ |
| P$< 3,2,5,5 >$ | P$< 3,7,5,5 >$ |
| P$< 4,2,5,5 >$ | P$< 4,7,5,5 >$ |
| P$< 5,2,5,5 >$ | P$< 5,7,5,5 >$ |
| P$< 6,2,5,5 >$ | P$< 6,7,5,5 >$ |
| P$< 7,2,5,5 >$ | P$< 7,7,5,5 >$ |
| P$< 8,2,5,5 >$ | P$< 8,7,5,5 >$ |
| R$< 1,1,5,5 >$ | R$< 1,8,5,5 >$ |
| R$< 8,1,5,5 >$ | R$< 8,8,5,5 >$ |
| N$< 2,1,5,5 >$ | N$< 2,8,5,5 >$ |
| N$< 7,1,5,5 >$ | N$< 7,8,5,5 >$ |
| B$< 3,1,5,5 >$ | B$< 3,8,5,5 >$ |
| B$< 6,1,5,5 >$ | B$< 6,8,5,5 >$ |
| Q$< 4,1,5,5 >$ | Q$< 4,8,5,5 >$ |
| K$< 5,1,5,5 >$ | K$< 5,8,5,5 >$ |

This is simply the starting position of 2-space chess with $5(\hat{x_3}+\hat{x_4})$ added to each piece. The choice of adding $5(\hat{x_3}+\hat{x_4})$ rather than $(\hat{x_3}+\hat{x_4})$ is in an attempt to mitigate biasing one region of the board and start the pieces in the "middle".